Fractional calculus of M-Series for Power Function

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Abstract— The present paper is to convert the M-series into M-series for Power function, using fractional calculus. This series is a particular case of H-function given by Inayat Hussain. The M-series is a powerful technique for solving the problems in a variety of fields such as in quantitative biology, scattering theory, signal processing and image processing etc.

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I. Introduction to the M-Series

The M-series is a particular case of the H- function of Inayat Hussain, [4]. A special role in the application of fractional calculus operators and in the solutions of fractional order differential equations. The Hypergeometric function and Mittag-Laffler function follow as its particular case [1], [5], [6]. Therefore, it is very interesting. We defined the M-series:

$$ {}_pM_q^\alpha\left(a_1\,.\,.\,.\,.\,a_p;,b_1\,.\,.\,.\,.\,b_q;z\right) = \sum_{k=o}^{\infty}\frac{(a_1)_k\,.\,.\,.\,.\,.\,(a_p)_k}{(b_1)_k\,.\,.\,.\,.\,.\,(b_q)_k}\frac{z^k}{\Gamma(\alpha k+1)} \qquad (1)$$

Here, $\alpha \epsilon$ C, $R(\alpha) > 0$, $(a_j)_k$ $(b_j)_k$ are pochammer symbols.

II. The M-Series for Power Function

Firstly, we give the notation and the definition of the M-series for power function with p upper parameters $a_1, a_2, \ldots\ldots a_p$ and q lower parameters $b_1, b_2, \ldots\ldots b_q$, defined by

$$ {}_pM_q^\alpha\left(a_1\,.\,.\,.\,.\,a_p;,b_1\,.\,.\,.\,.\,b_q;z^m\right) = {}_pM_q^\alpha(z^m)$$

$$ {}_pM_q^\alpha(z^m) = \sum_{k=o}^{\infty}\frac{(a_1)_k\,.\,.\,.\,.\,.\,(a_p)_k}{(b_1)_k\,.\,.\,.\,.\,.\,(b_q)_k}\frac{z^{mk}}{\Gamma(\alpha k+1)} \qquad (2)$$

Here, $\alpha\epsilon C$, $R(\alpha) > 0, m > 0$ and $(a_j)_k$ $(b_j)_k$ are pochammer symbols. The series (2) is defined when none of the denominator parameters $b_j s$, $j = 1,2, \ldots\ldots\ldots q$ is a negative integer or zero. If any parameter a_j is negative then the series (2) terminates into a polynomial in z. By using ratio test, it is evident that the series (2) is convergent for all z, when $q \geq p$, it is convergent for $|z| < 1$ when $p = q + 1$, divergent when $p > q + 1$. In some cases the series is convergent for $z = 1$, $z = -1$. Let us consider take,

$$\beta = \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j$$

when $p = q + 1$, the series is absolutely convergent for $|z| = 1$ if $R(\beta) < 0$, convergent for $z = -1$, if $0 \leq R(\beta) < 1$ and divergent for $|z| = 1$, if $1 \leq R(\beta)$.

Some special cases

A) ${}_0M_0^\alpha$ i.e. no upper or lower parameters and $m = 1$.

$$ {}_0M_0^\alpha(\ldots;\ldots;z) = \sum_{k=0}^{\infty}\frac{(z)^k}{\Gamma(\alpha k+1)} \qquad (3)$$

Thus the series reduced to the Mittag-Leffler function [1].

B) If $m = 1$, $\alpha = 1$ and no upper and lower parameters.

$$ {}_0M_0^1(\ldots;\ldots;z) = \sum_{k=0}^{\infty}\frac{(z)^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty}\frac{z^k}{k!} = e^z \qquad (4)$$

The series becomes the exponential series.

C) When $\alpha = 1$ and no lower parameter.

$$ {}_1M_0^\alpha(\alpha;\ldots;z^m) = \sum_{k=0}^{\infty}(\alpha)_k\frac{z^{mk}}{\Gamma(\alpha k+1)} = \sum_{k=0}^{\infty}(\alpha)_k\frac{z^{mk}}{k!} = (1-z^m)^k, \qquad (5)$$

for $|z| < 1$.

The series changes to the binomial series [6].

D) When $\alpha = 2$, $m = 2$ and no upper and lower parameters. We have,

$$ {}_0M_0^2(\ldots;\ldots;z^2) = \sum_{k=0}^{\infty}\frac{(z^2)^k}{\Gamma(2k+1)} = \sum_{k=0}^{\infty}\frac{z^{2k}}{2k!} = coshz \qquad (6)$$

Hence the series correspond to hyperbolic function.

E) When $\alpha = 1$, $m = 1$, we have,

$$ {}_pM_q^1\left(a_1\,.\,.\,.\,.\,a_p;,b_1\,.\,.\,.\,.\,b_q;z\right) = \sum_{k=o}^{\infty}\frac{(a_1)_k\,.\,.\,.\,.\,.\,(a_p)_k}{(b_1)_k\,.\,.\,.\,.\,.\,(b_q)_k}\frac{z^k}{\Gamma(k+1)} = {}_pF_q(z) \qquad (7)$$

Thus the series ${}_pM_q^1(z)$ reduces to the generalized hyper-geometric function [5], [6].

F) when $m = 1$, we have

$$ {}_pM_q^\alpha(z) = \sum_{k=o}^{\infty}\frac{(a_1)_k\,.\,.\,.\,.\,.\,(a_p)_k}{(b_1)_k\,.\,.\,.\,.\,.\,(b_q)_k}\frac{z^k}{\Gamma(\alpha k+1)} \qquad (8)$$

So, the series at $m = 1$ becomes the M-series [10].

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III. FRACTIONAL INTEGRAL AND FRACTIONAL DERIVATIVE OF THE M-SERIES FOR POWER FUNCTION

Let us consider the fractional Riemann-Liouville (R-L) integral operator [5] (for lower limit a=0 with respect to variable z) of the M-Series (2).

$$I_z^v\, {}_pM_q^\alpha\,(z^m) = \frac{1}{\Gamma(v)}\int_0^z (z-t)^{v-1}\, {}_pM_q^\alpha\,(t^m)\,dt$$

$$= \frac{1}{\Gamma(v)}\int_0^z (z-t)^{v-1}\sum_{k=o}^{\infty}\frac{(a_1)_k\,.....(a_p)_k}{(b_1)_k\,.....(b_q)_k}\frac{t^{mk}}{\Gamma(\alpha k+1)}\,dt$$

$$= \frac{1}{\Gamma(v)}\sum_{k=o}^{\infty}\frac{(a_1)_k\,.....(a_p)_k}{(b_1)_k\,.....(b_q)_k}\frac{1}{\Gamma(\alpha k+1)}\int_0^z (z-t)^{v-1}\,t^{mk}\,dt$$

$$= \frac{1}{\Gamma(v)}\sum_{k=o}^{\infty}\frac{(a_1)_k\,.....(a_p)_k}{(b_1)_k\,.....(b_q)_k}\frac{1}{\Gamma(\alpha k+1)}z^{mk+1+v-1}\,B(mk+1,v)$$

$$= \frac{1}{\Gamma(v)}\sum_{k=o}^{\infty}\frac{(a_1)_k\,.....(a_p)_k}{(b_1)_k\,.....(b_q)_k}\frac{1}{\Gamma(\alpha k+1)}\,z^{mk+v}\,\frac{\Gamma(mk+1)\Gamma(v)}{\Gamma(mk+1+v)}$$

$$= z^v\sum_{k=o}^{\infty}\frac{(a_1)_k\,.....(a_p)_k}{(b_1)_k\,.....(b_q)_k}\frac{z^{mk}}{\Gamma(\alpha k+1)}\,\frac{\Gamma(mk+1)}{\Gamma(mk+1+v)}$$

$$= \frac{\Gamma(mk+1-k)}{\Gamma(mk+1+v-k)}\,z^v\sum_{k=o}^{\infty}\frac{(a_1)_k\,.....(a_p)_k}{(b_1)_k\,.....(b_q)_k}\,\frac{(mk+1-k)_k}{(mk+1+v-k)_k}\frac{z^{mk}}{\Gamma(\alpha k+1)}$$

$$I_z^v\, {}_pM_q^\alpha\,(z^m) = \frac{\Gamma(mk+1-k)}{\Gamma(mk+1+v-k)}\,z^v\, {}_pM_q^\alpha$$
$$(a_1\,....a_p(mk+1-k);,b_1\,....b_{q,}\,(mk+1+v-k);\,z^m\,) \tag{9}$$

$R-L$ fractional derivative of M-Series which indices p,q are increased to $(p+1)(q+1)$.

Analogously, $R-L$ fractional derivative operator [6] of the M-Series with respect to z.

$$D_z^v\, {}_pM_q^\alpha\,(z^m) = \frac{1}{\Gamma(\mathrm{n}-v)}\left(\frac{d}{dz}\right)^n\int_0^z (z-t)^{n-v-1}\, {}_pM_q^\alpha\,(t^m)\,dt$$

$$D_z^v\, {}_pM_q^\alpha\,(z^m) = \frac{1}{\Gamma(\mathrm{n}-v)}\left(\frac{d}{dz}\right)^n\int_0^z (z-t)^{n-v-1}\sum_{k=o}^{\infty}\frac{(a_1)_k\,.....(a_p)_k}{(b_1)_k\,.....(b_q)_k}\frac{t^{mk}}{\Gamma(\alpha k+1)}dt$$

$$= \frac{1}{\Gamma(\mathrm{n}-v)}\sum_{k=o}^{\infty}\frac{(a_1)_k\,.....(a_p)_k}{(b_1)_k\,.....(b_q)_k}\frac{1}{\Gamma(\alpha k+1)}\left(\frac{d}{dz}\right)^n\int_0^z (z-t)^{n-v-1}\,t^{mk}\,dt$$

$$= \frac{1}{\Gamma(\mathrm{n}-v)}\sum_{k=o}^{\infty}\frac{(a_1)_k\,.....(a_p)_k}{(b_1)_k\,.....(b_q)_k}\frac{1}{\Gamma(\alpha k+1)}\left(\frac{d}{dz}\right)^n z^{mk+n-v}\,B(mk+1,n-v)$$

We use the modified Beta-function:

$$\int_a^b (b-t)^{\beta-1}\,(t-a)^{\alpha-1}\,dt\; = (b-a)^{\alpha+\beta-1}\,B(\alpha,\beta),$$

for $R(\alpha)>0, R(\beta)>0$

$$= \frac{1}{\Gamma(\mathrm{n}-v)}\sum_{k=o}^{\infty}\frac{(a_1)_k\,.....(a_p)_k}{(b_1)_k\,.....(b_q)_k}\frac{1}{\Gamma(\alpha k+1)}\left(\frac{d}{dz}\right)^n z^{mk+n-v}\frac{\Gamma(mk+1)\Gamma(\mathrm{n}-v)}{\Gamma(mk+1+n-v)} \tag{10}$$

Where $k+1>0, n-v>0$

Differentiation n times the term z^{mk+n-v} and using again $\Gamma(a+k)=(a)_k\Gamma(a)$, representation (9) reduces to

$$= \sum_{k=o}^{\infty}\frac{(a_1)_k\,.....(a_p)_k}{(b_1)_k\,.....(b_q)_k}\frac{(mk+n-v)!}{\Gamma(\alpha k+1)}z^{mk-v}\frac{\Gamma(mk+1)}{\Gamma(mk+1+n-v)}$$

$$= z^{-v}\sum_{k=o}^{\infty}\frac{(a_1)_k\,.....(a_p)_k}{(b_1)_k\,.....(b_q)_k}\frac{z^{mk}}{\Gamma(\alpha k+1)}\frac{\Gamma(mk+1)}{\Gamma(mk-v+1)}$$

$$= z^{-v}\sum_{k=o}^{\infty}\frac{(a_1)_k\,.....(a_p)_k\,(mk+1-k)_k}{(b_1)_k\,.....(b_q)_k(mk-v+1-k)_k}\,\frac{\Gamma(mk+1-k)}{\Gamma(mk-v+1-k)}\;z^{mk}$$

$$D_z^v\, {}_pM_q^\alpha\,(z^m) = \frac{\Gamma(mk+1-k)}{\Gamma(mk-v+1-k)}z^{-v}\;{}_{p+1}M_{q+1}^\alpha$$
$$(a_{1.\,..}a_p,(mk+1-k);b_{1.\,..}b_q;(mk-v+1-k);z^m) \tag{11}$$

$(mk+1)>0$, gives a $R-L$ fractional derivative of M-series power function, which indices p, q are increased to (p+1),(q+1).

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REFERENCES

[1] G.M. Mittag-Leffler, Sur la nouvelle function $E_\alpha(x)$. C. R. Acad, Sci. Paris (Ser. II) 137 (1903) 554-558.

[2] C. Fox, The G and H- function as symmetrical Fourier kernels. Trans. Amer. Math. Soc. 98 (1961), 395-429.

[3] A. M. Mathai, R. K. Saxena, Tha H-function with Application in Statistics and Other Disciplines. John Wiley and Sons, Inc., New York (1978).

[4] A. A. Inayat Hussain, New properties of hypergeometric series derivable from Feynman integrals, II: A generalization of H −function. J. Phys. A: Math. Gen. 20 (1987), 4119-4128.

[5] A. P. Prudnikov, Yu. BBrychkov, O. I. Marichev, Integrals and Series. Vol. 3 : More Special Functions. Gordon and Breach, New York NJ (1990).

[6] S. Samko, A. Kilbas, O. Marichev, Fractional Integrals and derivaties, Theory and Applications. Gordon and Breach, New York (1993).

[7] I. Podlubny, Fractional Differential Equations. Acad. Press, San Diego-New, York (1999).

[8] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations. Elsevier, North Holland Math. Studies 204, Amsterdam, etc. (2006).

[9] V. Kiryakova , some special functions related to fractional calculus and fractional (non-integer) order control systems and equations. Facta Universitatis (Sci. J. of Univ. Nis) Automatic Control and Robotics, 7 No.1 (2008), 79-98.

[10] Sharma, M.: Fractional Integration and Fractional Differentiation of the M-Series. J. Fract. Calc. and Appl. Anal. Vol. 11, No. 2 (2008), 187-191.

[11] Sharma, M. and Jain, R.: A note on a generalized M-Series as a special function of fractional calculus. J. Fract. Calc. and Appl. Anal. Vol. 12, No. 4 (2009), 449-452.

[12] R. K. Sexena, A remark on a paper on M-series. Fract. Calc. Appl. Anal. 12, No.1 (2009), 109-110.

[13] V. Kiryakova, The special functions of fractional calculus as generalized fractional calculus operators of some basic functions. Computers and Math. with Appl. 59 (2010).

[14] Abramowitz, M., Stegun, I. A.: Handbook of Mathematical functions. Dover (1965), New York.